Utility Optimal Coding for Packet Transmission over Wireless Networks – Part II: Networks of Packet Erasure Channels

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Abstract*—We define a class of multi–hop erasure networks that approximates a wireless multi–hop network. The network carries unicast flows for multiple users, and each information packet within a flow is required to be decoded at the flow destination within a specified delay deadline. The allocation of coding rates amongst flows/users is constrained by network capacity. We propose a proportional fair transmission scheme that maximises the sum utility of flow throughputs. This is achieved by *jointly optimising the packet coding rates and the allocation of bits of coded packets across transmission slots.

***Index Terms*—Code rate selection, cross layer optimisation, network utility maximisation, packet erasure channels, scheduling**

I. Introduction

In a communication network, the network capacity is shared by a set of flows. There is a contention for resources among the flows, which leads to many interesting problems. One such problem, is *how to allocate the resources optimally across the (competing) flows, when the physical layer is erroneous*. Specifically, schedule/transmit time for a flow is a resource that has to be optimally allocated among the competing flows. In this work, we pose a network utility maximisation problem subject to scheduling constraints that solve a resource allocation problem. In another work, we studied the problem of optimal resource allocation in networks [1].

We define a class of multi–hop erasure networks, and consider packet communication over this class. The network consists of a set of $C \geq 1$ cells $\mathcal{C} = \{1, 2, \cdots, C\}$ which define the "interference domains" in the network. We allow intra–cell interference (*i.e* transmissions by nodes within the same cell interfere) but assume that there is no inter–cell interference. This captures, for example, common network architectures where nodes within a given cell use the same radio channel while neighbouring cells using orthogonal radio channels. Within each cell, any two nodes are within the decoding range of each other, and hence, can communicate with each other. The cells are interconnected using multi–radio bridging nodes to create a multi–hop wireless network. A multi–radio bridging node i connecting the set of cells $\mathcal{B}(i) = \{c_1, .., c_n\} \subset \mathcal{C}$ can be thought of as a set of n single radio nodes, one in each cell, interconnected by a high–speed, loss–free wired backplane (see Figure 1).

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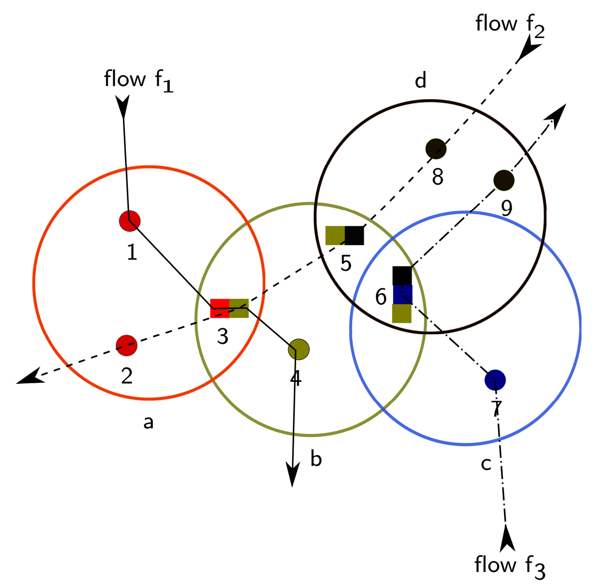

Fig. 1. **An illustration of a wireless mesh network with 4 cells.** Cells a, b, c, and d use orthogonal channels CH_1, CH_2, CH_3, and CH_4 respectively. Nodes 3, 5, and 6 are *bridge nodes*. The bridge node 3 (resp. 5 and 6) is provided a time slice of each of the channels CH_1 & CH_2 (resp. CH_2 & CH_4 for node 5 and CH_2& CH_3& CH_4 for node 6). Three flows f_1, f_2, and f_3 are considered. In this example, $\mathcal{C}_{f_1} = \{a, b\}$, $\mathcal{C}_{f_2} = \{d, b, a\}$, and $\mathcal{C}_{f_3} = \{c, d\}$.

Data is transmitted across this multi–hop network as a set $\mathcal{F} = \{1, 2, \cdots, F\}$, $F \geq 1$ of unicast flows. The route of each flow $f \in \mathcal{F}$ is given by $\mathcal{C}_f = \{c_1(f), c_2(f), \cdots, c_{\ell_f}(f)\}$, where the source node $s(f) \in c_1(f)$ and the destination node $d(f) \in c_{\ell_f}(f)$. We assume loop–free flows (i.e., no two cells in $\mathcal{C}_f$ are same). Figures 1 and 2 illustrate this network setup. A scheduler assigns a time slice of duration $T_{f,c} > 0$ time units to each flow f that flows through cell c, subject to the constraint that $\sum_{f:c\in\mathcal{C}_f} T_{f,c} \leq T_c$ where T_c is the period of the schedule in cell c. We consider a periodic scheduling strategy (see Figure 2) in which, in each cell c, service is given to the flows in a round robin fashion, and that each flow f in cell c gets a time slice of $T_{f,c}$ units in every schedule.

The scheduled transmit times for flow f in source cell $c_1(f)$ define time slots for flow f. We assume that a new information packet arrives in each time slot, which allows us to simplify the analysis by ignoring queueing. Information packets of each flow f at the source node $S(f)$ consist of a block of k_f

symbols. Each packet of flow f is encoded into codewords of length $n_f = k_f/r_f$ symbols, with coding rate $0 < r_f \leq 1$. The code employed for encoding is discussed in Section II. We require sufficient transmit times at each cell along route $\mathcal{C}_f$ to allow n_f coded symbols to be transmitted in every schedule period. Hence there is no queueing at the cells along the route of a flow. It is not apparent at this point whether it is optimal for flow f to transmit a single code–word of n_f symbols or transmit a block of n_f symbols where each block carries some portions of each of a set of coded packets.

Channel Model: The channel in cell c for flow f is considered to be a packet erasure channel with the probability of packet erasure being $\beta_{f,c} \in [0,1]$. Thus, the end–to–end channel for flow f is a packet erasure channel with the probability of packet erasure being

$$\beta_f = 1 - \prod_{c \in \mathcal{C}_f} [1 - \beta_{f,c}]$$

Let the Bernoulli random variable $E_f[i]$ indicate the end–to–end erasure seen by the ith block of flow f (independent of the erasure seen by other blocks) of flow f. Note that $E_f[i] = 1$ means that the ith block is erased, and $E_f[i] = 0$ means that the ith block is received successfully. Note that $\mathsf{P}\{E_f[i] = 1\} = \beta_f = 1 - \mathsf{P}\{E_f[i] = 0\}$.

Each packet has a deadline of D_f slots, by which time it must be decoded. Such a delay constraint is natural in applications such as video streaming. A packet is in error if the destination fails to decode the packet by the deadline. Letting $e_f(r_f)$ denote the error probability that a packet fails to be decoded before its deadline, the expected number of information symbols successfully received is $S_f(r_f) = k_f(1 - e_f(r_f))$. Other things being equal, we expect that decreasing r_f (i.e., increasing the number of coded symbols $n_f = k_f/r_f$ sent) decreases error probability e_f and so increases S_f. However, since the network capacity is limited, and is shared by multiple flows, increasing the coded packet size n_{f_1} of flow f_1 generally requires decreasing the packet size n_{f_2} for some other flow f_2. That is, increasing S_{f_1} comes at the cost of decreasing S_{f_2}. We are interested in understanding this trade–off, and in analysing the optimal fair allocation of coding rates amongst users/flows.

Our main contribution is the analysis of fairness in the allocation of coding rates between users/flows competing for limited network capacity. In particular, we adopt a utility–fair framework, and propose a scheme for obtaining the proportional fair allocation of coding rates, *i.e.* the allocation of coding rates that maximises $\sum_{f \in \mathcal{F}} \log S_f(r_f)$ subject to network capacity constraints. This problem, which we show in Section III, requires solving a non–convex optimisation problem. Specifically, at the physical layer, the (channel) coding rate of a flow can be lowered (to alleviate its channel errors) only at the expense of increasing the coding rates of other flows. Also, at the network layer, the length of schedules of each flow should be chosen in such a way that it maximises the network utility. Interestingly, we show in our problem formulation that the coding rate and the scheduling are tightly coupled. Also, we show that for a $\log$ (network) utility function (which typically gives proportional fair allocation of resources)

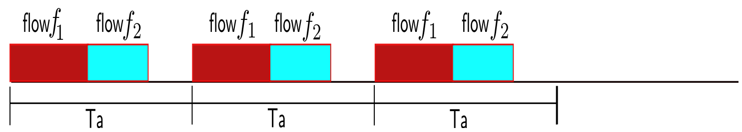

Fig. 2. **An illustration of transmission scheme in cell a of the network in Figure 1:** Every transmission schedule of T_a time units is time–shared by nodes 1 and 3. Note that $\phi_\Delta(f) N_f R_f$ symbols of the encoded packet p are transmitted in transmission schedule $p+\Delta$, where $\Delta \in \{0, 1, 2, \cdots, n_f - 1\}$. The scheduling or capacity constraint of cell a may not be tight (indicated by empty time slice in the figure), as the rates of flows f_1 and f_2 are governed by the whole network.

the optimum rate allocation (in general) gives unequal air–times which is quite different from the previously known result of proportional fair allocation being the same as that of equal air–time allocation ([2]). This problem, which we show in Section III, requires solving a non–convex optimisation problem. Our work differs from the previous work on network utility maximisation (see [3] and the references therein) in the following manner. To the best of our knowledge, this is the first work that computes the optimal coding rate for a given scheduling (or capacity) constraints in the utility–optimal framework.

The rest of the paper is organised as follows. In Section II, we obtain a measure for the end–to–end packet erasure, and describe the throughput of the network. We then formulate a network utility maximisation problem subject to constraints on the transmission schedule lengths. In Section III, we obtain the optimum transmission strategy and the optimum packet–level coding rates for each flow in the network. In Section V, we provide some simple examples to illustrate our results. Due to lack of space, the proofs of various Lemmas are omitted.

II. Problem Formulation

The encoding has two stages. The first stage is the encoding of each information packet using a standard generator matrix such as a Reed–Solomon code or a fountain code [4]. Let $P_f[t]$ denote the information packet that arrives at the source of flow f in slot t. A packet $P_f[t]$ of flow f has k_f symbols, the encoded packet $C_f[t]$ of which is of size $n_f = k_f/r_f$ with $0 < r_f \leq 1$, and we assume that the code is such that the packet $P_f[t]$ can be reconstructed from *any* of its k_f encoded symbols (this is possible, for example, by Reed–Solomon codes).

The second stage allocates the content of the encoded packet C_t of the first stage across the *transmitted* packets. Each encoded packet is segmented into D_f portions (where we recall that D_f is the decoding deadline requirement for each packet of flow f), the size of the Δth portion being $\phi_f(\Delta) n_f$, where $\Delta \in \{0, 1, \cdots, D_f - 1\}$ and $0 \leqslant \phi_f(\Delta) \leqslant 1$. At transmission slot t, a transmitted packet is assembled from the $\phi_f(0)$ portion of $C_f[t]$, the $\phi_f(1)$ portion of $C_f[t-1]$, and so on until the $\phi_f(D_f - 1)$th portion of packet $C_f[t - D_f + 1]$. This procedure is illustrated in Figure 3 for $n_f = 3$. Note that the transmitted packet is of size n_f symbols. To decode a packet $P_f[t]$ of flow f, we use the transmitted packets that are received during the transmission slots $t, t+1, \cdots, t + D_f - 1$. Note that the conventional strategy of transmitting an encoded packet every transmission slot corresponds to the special case:

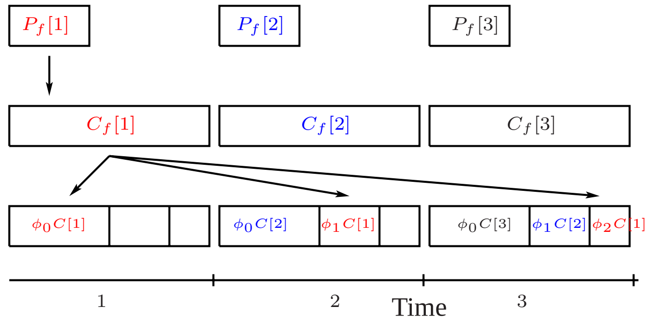

Fig. 3. Two stage encoding (example of $D_f = 3$): information packet $P_f[1]$ of size k_f is encoded to $C_f[1]$ of size $n_f = k_f/r_f$, the contents of which are allocated across subpackets $\phi_0 C_f[1], \phi_1 C_f[1], \phi_2 C_f[1]$ across 3 timeslots.

$\phi_f(0) = 1$ and $\phi_f(1) = \phi_f(2) = \cdots = \phi_f(D_f - 1) = 0$. We call the transmission scheme outlined above with general $\phi_.(\Delta)$s a *generalised block transmission scheme.*

A. Network Constraints on Coding Rate

Let $w_{f,c}$ be the PHY rate of transmission of flow f in cell c. For each transmitted packet of flow f, each cell $c \in \mathcal{C}_f$ along its route must allocate at least $\frac{n_f}{w_{f,c}}$ units of time to transmit the packet (or encoded block). Let $\mathcal{F}_c := \{f \in \mathcal{F} : c \in \mathcal{C}_f\}$ be the set of flows that are routed through cell c. We recall that the transmissions in any cell c are scheduled in a TDMA fashion, and hence, the total time required for transmitting packets for all flows in cell c is given by $\sum_{f \in \mathcal{F}_c} \frac{n_f}{w_{f,c}}$. Since, for cell c, the transmission schedule interval is T_c units of time, the coding rates r_f must satisfy the schedulability constraint $\sum_{f \in \mathcal{F}_c} \frac{n_f}{w_{f,c}} \leqslant T_c$.

B. Error Probability – Upper bound

Lemma 1. *The end–to–end probability of a packet erasure for flow f is bounded by*

$$\begin{aligned}
&\widetilde{e}_f \\
&= \mathsf{P}\left\{\sum_{\Delta=0}^{D_f-1} \phi_f(\Delta)\frac{k_f}{r_f}E_f[\Delta] > n_f - k_f\right\} \\
&\leq \exp\left(-\left[\theta_f(1-r_f) - \sum_{\Delta=0}^{D_f-1} \ln\left(1-\beta_f+\beta_f e^{\theta_f\cdot\phi_f(\Delta)}\right)\right]\right) \\
&=: e_f.
\end{aligned}$$

where $\theta_f > 0$ is the Chernoff–bound parameter.

Let the random variable $\alpha_f[t]$ indicate whether packet $P_f[t]$ is successfully decoded or not, i.e.,

$$\alpha_f[t] = \begin{cases} 1, & \text{if packet } P_f[t] \text{ is decoded successfully} \\ 0, & \text{otherwise.} \end{cases}$$

We note here that the decoding errors for the successive packets are correlated, as each encoded packet overlaps with the transmission of previous $D_f - 1$ packets and the successive $D_f - 1$ packets. Hence, the sequence of random variables $\alpha_f[1], \alpha_f[2], \alpha_f[3], \cdots$ are correlated. But, the probability of any $\alpha_f[t] = 0$ is upper bounded by Lemma 1.

III. Network Utility Maximisation

For flow f, the total expected throughput as a result of transmitting $T \geq 1$ packets is given by

$$\begin{aligned}
&S_f(T) \\
&= \sum_{(x_1,x_2,\cdots,x_T)\in\{0,1\}^T} \left(\sum_{t=1}^{T} k_f x_t\right) \mathsf{P}\left\{\alpha_f[t] = x_t, t = 1,2,\cdots,T\right\}
\end{aligned}$$

Note that the joint probability mass function $\mathsf{P}\{\alpha_f[t] = x_t, t = 1, 2, \cdots, T\}$ is not a product–form distribution as the packet erasures $\alpha_f[t]$s are correlated. However, the above expectation can be written as

$$\begin{aligned}
S_f(T) &= \sum_{t=1}^{T} \sum_{x_t\in\{0,1\}} k_f x_t \mathsf{P}\{\alpha_f[t] = x_t\} \\
&= T \cdot k_f \cdot (1 - e_f)
\end{aligned}$$

Thus, the (average expected) flow throughput is defined as

$$\begin{aligned}
S_f &= \lim_{T\to\infty} \frac{S_f(T)}{T} \\
&= k_f \cdot (1 - e_f).
\end{aligned}$$

We are interested in maximising the utility of the network which is defined as the sum utility of flow throughputs. We consider the log of throughput as the candidate for the utility function being motivated by the desirable properties like proportional fairness that it possesses.

We define the following notations: the Chernoff–bound parameters $\boldsymbol{\theta} := [\theta_f]_{f\in\mathcal{F}}$, coding rates $\boldsymbol{r} := [r_f]_{f\in\mathcal{F}}$, and the allocation of coded bits across transmission slots $\boldsymbol{\Phi} := [\boldsymbol{\phi}_f]_{f\in\mathcal{F}}$ where $\boldsymbol{\phi}_f := [\phi_f(0), \phi_f(1), \cdots, \phi_f(D_f - 1)]$. Thus, we define the network utility as

$$\begin{aligned}
\widetilde{U}(\boldsymbol{\Phi},\boldsymbol{\theta},\boldsymbol{r}) &:= \sum_{f\in\mathcal{F}} \ln\left(k_f\left(1 - e_f(\boldsymbol{\phi}_f,\theta_f,r_f)\right)\right) \\
&=: \sum_{f\in\mathcal{F}} \ln(k_f) + U(\boldsymbol{\Phi},\boldsymbol{\theta},\boldsymbol{r}) \qquad (1)
\end{aligned}$$

The problem is to obtain the optimum coded bit allocation $\boldsymbol{\Phi}^*$, the optimum Chernoff–bound parameter $\boldsymbol{\theta}^*$, and the optimum coding rate $\boldsymbol{r}^*$ that maximises the network utility. Since, k_f, the size of information packets of each flow f is given, maximising the network utility is equivalent to maximising $U(\boldsymbol{\Phi},\boldsymbol{\theta},\boldsymbol{r}) := \sum_{f\in\mathcal{F}} \ln(1 - e_f)$. Thus, we define the following problem

P1:

$$\begin{aligned}
&\max_{\boldsymbol{\Phi},\boldsymbol{\theta},\boldsymbol{r}} \quad U(\boldsymbol{\Phi},\boldsymbol{\theta},\boldsymbol{r}) \\
&\text{subject to} \quad \sum_{f:c\in\mathcal{C}_f} \frac{k_f}{r_f w_{f,c}} \leq T_c, \qquad \forall c \in \mathcal{C} \qquad (2) \\
&\qquad \sum_{\Delta=0}^{D_f-1} \phi_f(\Delta) = 1, \qquad \forall f \in \mathcal{F} \qquad (3) \\
&\qquad \phi_f(\Delta) \geq 0, \qquad \forall f \in \mathcal{F}, 0 \leq \Delta \leq D_f - 1 \\
&\qquad \theta_f > 0, \qquad \forall f \in \mathcal{F} \\
&\qquad r_f \leq \overline{\lambda}_f \qquad \forall f \in \mathcal{F} \\
&\qquad r_f \geq \underline{\lambda}_f \qquad \forall f \in \mathcal{F}
\end{aligned}$$

We note that the Eqn. (2) enforces the network capacity (or the network schedulability) constraint. The objective function $U(\mathbf{\Phi}, \boldsymbol{\theta}, \mathbf{r})$ is separable in $(\boldsymbol{\phi}_f, \theta_f, r_f)$ for each flow f. Importantly, the component of utility function for each flow f given by $\ln(1 - e_f(\boldsymbol{\phi}_f, \theta_f, r_f))$ is not jointly concave in $(\boldsymbol{\phi}_f, \theta_f, r_f)$. However, $\ln(1 - e_f(\boldsymbol{\phi}_f, \theta_f, r_f))$ is concave in each of $\phi_f(\cdot)$, θ_f, and r_f. Hence, the network utility maximisation problem $\mathbf{P1}$ is not in the standard convex optimisation framework. Instead, we pose the following problem,

P2:

$$\max_{\mathbf{\Phi}} \max_{\boldsymbol{\theta}} \max_{\boldsymbol{r}} \sum_{f \in \mathcal{F}} \ln(1 - e_f(\boldsymbol{\phi}_f, \theta_f, r_f)) \tag{4}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{f: c \in \mathcal{C}_f} \frac{k_f}{r_f w_{f,c}} \le T_c, && \forall c \in \mathcal{C} \\
& \sum_{\Delta=0}^{D_f-1} \phi_f(\Delta) = 1, && \forall f \in \mathcal{F} \\
& \phi_f(\Delta) \ge 0, && \forall f \in \mathcal{F}, 0 \le \Delta \le D_f - 1 \\
& \theta_f > 0, && \forall f \in \mathcal{F} \\
& r_f \le \overline{\lambda}_f && \forall f \in \mathcal{F} \\
& r_f \ge \underline{\lambda}_f && \forall f \in \mathcal{F}
\end{aligned}$$

In general, the solution to $\mathbf{P2}$ need not be the solution to $\mathbf{P1}$. However, in our problem, we show that $\mathbf{P2}$ achieves the solution of $\mathbf{P1}$.

Lemma 2. *. For a function $f : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ that is concave in y and in z, but not jointly in (y, z), the solution to the joint optimisation problem for convex sets $\mathcal{Y}$ and $\mathcal{Z}$*

$$\max_{y \in \mathcal{Y}, z \in \mathcal{Z}} f(y, z) \tag{5}$$

is the same as

$$\max_{z \in \mathcal{Z}} \max_{y \in \mathcal{Y}} f(y, z), \tag{6}$$

if $f(y^*(z), z)$ is a concave function of z, where for each $z \in \mathcal{Z}$, $y^*(z) := \arg\max_{y \in \mathcal{Y}} f(y, z)$.

We note that for each r_f and θ_f, the probability of error e_f is convex in $\boldsymbol{\phi}_f$, and hence, $\ln(1 - e_f)$ is concave in $\boldsymbol{\phi}_f$. Thus, we first solve for the optimum code bit allocation $\boldsymbol{\phi}_f^*$ in Section IV-A. Then, using the optimum code bit allocation, we solve for the optimum Chernoff bound parameter $\boldsymbol{\theta}^*$ which we describe in subsection IV-B. After having solved for the optimum $\boldsymbol{\theta}^*$, we show in Section IV-C that $U(\mathbf{\Phi}^*, \boldsymbol{\theta}^*(\boldsymbol{r}), \boldsymbol{r})$ is a concave function of $\boldsymbol{r}$. Hence, from Lemma 2, the solution to problem $(\mathbf{P2})$ (the maximisation problem that separately obtains the optimum $\boldsymbol{\theta}^*$ and optimum $\boldsymbol{r}^*$) is globally optimum. We study the rate optimisation problem that obtains $\boldsymbol{r}^*$ in subsection IV-D.

IV. Utility Optimum Rate Allocation

A. Optimal Code Bit Allocation Φ

We consider the maximisation problem defined in Eqn. 4 for a given coding rate vector $\boldsymbol{r}$ and Chernoff–bound parameter vector $\boldsymbol{\theta}$, and obtain the optimum $\boldsymbol{\phi}_f$ for each flow $f \in \mathcal{F}$. The sub–problem is given by

$$\begin{aligned}
\max_{\boldsymbol{\phi}_f} \quad & \sum_{f \in \mathcal{F}} \ln(1 - e_f(\boldsymbol{\phi}_f, \theta_f, r_f)) \\
\text{subject to} \quad & \sum_{\Delta=0}^{D_f-1} \phi_f(\Delta) \;=\; 1, \quad \forall f \in \mathcal{F} \\
& \phi_f(\Delta) \;\ge\; 0, \quad \forall f \in \mathcal{F}, \forall \Delta \le D_f - 1.
\end{aligned}$$

This is a separable convex optimisation problem, and hence can be solved by Lagrangian method. Let μ_f be a Lagrangian multiplier for the constraint $\sum_{\Delta=0}^{D_f-1} \phi_f(\Delta) = 1$, and define $\boldsymbol{\mu} = [\mu_f]_{f \in \mathcal{F}}$. The Lagrangian function is given by

$$L(\mathbf{\Phi}, \boldsymbol{\mu}) = \sum_{f \in \mathcal{F}} \ln(1 - e_f) - \sum_{f \in \mathcal{F}} \mu_f \left(1 - \sum_{\Delta=0}^{D_f-1} \phi_f(\Delta)\right)$$

Applying KKT condition,

$$\frac{\partial L}{\partial \phi_f(i)} \Big|_{\phi_f(i)^*} \;=\; 0,$$

we get

$$\begin{aligned}
0 \;&=\; \frac{-e_f}{1 - e_f} \cdot \frac{\beta_f \theta_f e^{\theta_f \phi_f^*(i)}}{1 - \beta_f + \beta_f e^{\theta_f \phi_f^*(i)}} + \mu_f \\
\text{or, } e^{\theta_f \phi_f^*(i)} \;&=\; \frac{1 - \beta_f}{\beta_f} \frac{(1 - e_f)\mu_f}{\theta_f e_f - \mu_f (1 - e_f)}
\end{aligned} \tag{7}$$

for $i = 0, 1, 2, \cdots, n_f - 1$. Since, the RHS of Eqn. 7 is the same for all i, we get $\phi_f^*(i) = \phi_f^*(j)$, and hence

$$\phi_f^*(\Delta) = \frac{1}{D_f}, \; \forall \Delta = 0, 1, \cdots, D_f - 1.$$

Thus, $\mathbf{\Phi}^*$ allocates equal portions of an encoded packet across transmission schedules with a delay of $0, 1, \cdots, D_f - 1$, unlike the conventional transmission scheme which transmits all the coded bits of a packet in one shot. Hence, $e_f(\boldsymbol{\phi}_f^*, \theta_f, r_f)$ is

$$e_f = \exp\left(-\left[\theta_f(1 - r_f) - D_f \ln\left(1 - \beta_f + \beta_f e^{\frac{\theta_f}{D_f}}\right)\right]\right). \tag{8}$$

B. Optimal θ^*

We now consider the optimum Chernoff–bound parameter problem with the optimum coded bits allocation $\mathbf{\Phi}^*$, and for any given coding rate vector $\boldsymbol{r} \in [\underline{\lambda}_f, \overline{\lambda}_f]^F$.

$$\begin{aligned}
\max_{\boldsymbol{\theta}} \; & \sum_{f \in \mathcal{F}} \ln\left(1 - e_f(\boldsymbol{\phi}_f^*, \theta_f, r_f)\right) \\
\text{subject to} \quad & \theta_f > 0, \qquad \forall f \in \mathcal{F}
\end{aligned} \tag{9}$$

We note that the objective function is separable in θ_fs, and that e_f is convex in θ_f. Hence, the problem defined in Eqn. (9), is a concave maximisation problem. The partial derivative of e_f with respect to θ_f is given by

$$\frac{\partial e_f}{\partial \theta_f} = -e_f \cdot \left[(1 - r_f) - \frac{\beta_f e^{\theta_f / D_f}}{1 - \beta_f + \beta_f e^{\theta_f / D_f}}\right].$$

Observe that $\frac{\beta_f e^{\theta_f/D_f}}{1-\beta_f+\beta_f e^{\theta_f/D_f}}$ is an increasing function of θ_f. Thus, if, for $\theta_f = 0$, $1 - r_f - \frac{\beta_f}{1-\beta_f+\beta_f} < 0$ or $r_f > 1 - \beta_f$, the derivative is positive for all $\theta_f > 0$, or e_f is an increasing function of θ_f. Hence, for $r_f > 1 - \beta_f$, the optimum θ_f^* is arbitrarily close to 0 which yields e_f arbitrarily close to 1. Thus, for error recovery, for any end–to–end error probability β_f, the coding rate should be smaller than $1 - \beta_f$, in which case, we obtain the optimum θ_f^* by equating the partial derivative of e_f with respect to θ_f to zero.

$$\begin{aligned}
\text{i.e.,} \quad \frac{\beta_f e^{\theta_f^*/D_f}}{1-\beta_f+\beta_f e^{\theta_f^*/D_f}} &= 1 - r_f \\
\text{or,} \quad e^{\theta_f^*/D_f} &= \frac{1-r_f}{\beta_f}\frac{1-\beta_f}{r_f} \\
\text{or,} \quad \theta_f^* &= D_f\left[\ln\left(\frac{1-r_f}{\beta_f}\right) - \ln\left(\frac{r_f}{1-\beta_f}\right)\right].
\end{aligned}$$

Thus, the probability of a packet decoding error for a given r_f with the optimum allocation of coded bits $\mathbf{\Phi}^*$, and the optimum Chernoff–bound parameter θ_f^*, is

$$\begin{aligned}
& e_f \\
&= \exp\left(-D_f\left[(1-r_f)\ln\left(\frac{1-r_f}{\beta_f}\right) + r_f \ln\left(\frac{r_f}{1-\beta_f}\right)\right]\right) \\
&= \exp\left(-D_f \cdot \mathrm{KL}(\mathcal{B}(1-r_f)||\mathcal{B}(\beta_f))\right)
\end{aligned}$$

where $\mathrm{KL}(f_1, f_2)$ is the Kullback–Leibler divergence between the probability mass functions (pmfs) f_1 and f_2.

C. A convex optimisation framework to obtain optimal r_f^*

If $\ln(1 - e_f(\boldsymbol{\phi}_f^*, \theta_f^*, r_f))$ is concave in r_f, then one can obtain the optimum r_f^* using convex optimisation framework. To show the concavity of $\ln(1-e_f(\boldsymbol{\phi}_f^*, \theta_f^*, r_f))$ it is sufficient to show that $e_f(\boldsymbol{\phi}_f^*, \theta_f^*, r_f)$ is convex in r_f. Note that

$$\begin{aligned}
\frac{\partial e_f}{\partial r_f} &= e_f \cdot \theta_f^*(r_f) \\
\frac{\partial^2 e_f}{\partial r_f^2} &= e_f\left[\theta_f^{*2} - \frac{D_f}{r_f(1-r_f)}\right]
\end{aligned}$$

e_f is convex if

$$\left[\ln\left(\frac{1-r_f}{\beta_f}\right) - \ln\left(\frac{r_f}{1-\beta_f}\right)\right]^2 \geq \frac{D_f}{r_f(1-r_f)},$$

or,

$$\ln\left(\frac{1-r_f}{r_f}\frac{1-\beta_f}{\beta_f}\right) \geq \frac{\sqrt{D_f}}{\sqrt{r_f(1-r_f)}}$$

$$\text{or,} \quad \frac{\sqrt{D_f}}{\sqrt{r_f(1-r_f)}} - \ln\left(\frac{1-r_f}{r_f}\frac{1-\beta_f}{\beta_f}\right) \leq 0$$

The function $\frac{1}{\sqrt{r_f(1-r_f)}}$ is convex in r_f. Also, $\ln\left(\frac{1-r_f}{r_f}\right)$ is decreasing with r_f, and hence, $-\ln\left(\frac{1-r_f}{r_f}\frac{1-\beta_f}{\beta_f}\right) \leq -\ln\left(\frac{1-\overline{\lambda}_f}{\overline{\lambda}_f}\frac{1-\beta_f}{\beta_f}\right)$. Thus, we have a sufficient condition

$$\frac{\sqrt{D}_f}{\sqrt{r_f(1-r_f)}} - \ln\left(\frac{1-\overline{\lambda}_f}{\overline{\lambda}_f}\frac{1-\beta_f}{\beta_f}\right) \leq 0 \tag{10}$$

The above condition requires the delay deadline D_f to be smaller than some $\overline{D}_f(r_f)$. We consider D_fs to satisfy this condition, and hence, the rate optimisation problem is a concave maximisation problem. For the sake of completeness, we include this as a constraint in the problem formulation. However, this condition is not an active constraint.

D. Optimal Coding Rate r

From the previous subsection, we observe under the delay constraint Eqn. (10) that $e_f(\boldsymbol{\phi}_f^*, \theta_f^*(r_f), r_f)$ is convex in r_f, and hence, we obtain the optimum coding rate r_f^* using convex optimisation method. Also, from Lemma 2, it is clear that r_f^* is the unique globally optimum rate. Thus, we solve the following network utility maximisation problem

$$\max_{\boldsymbol{r}} \sum_{f\in\mathcal{F}} \ln\left(1 - e_f(\boldsymbol{\phi}_f^*, \theta_f^*(r_f), r_f)\right) \tag{11}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{f:c\in\mathcal{C}_f} \frac{k_f}{r_f w_{f,c}} \leq T_c, && \forall c \in \mathcal{C} \\
& r_f \leq \overline{\lambda}_f && \forall f \in \mathcal{F} \\
& r_f \geq \underline{\lambda}_f && \forall f \in \mathcal{F} \\
& \frac{\sqrt{D_f}}{\sqrt{r_f(1-r_f)}} - a \leq 0 && \forall f \in \mathcal{F} \quad (12)
\end{aligned}$$

where $a = \ln\left(\frac{1-\overline{\lambda}_f}{\overline{\lambda}_f}\frac{1-\beta_f}{\beta_f}\right)$. It is clear that the objective function is separable and concave, and hence, can be solved using Lagrangian relaxation method. Also, we note here that the constraint represented by Eqn. (12) is not an active constraint, and hence, there is no Lagrangian cost to this constraint. We note here that the coding rate should be such that k_f/r_f is an integer, and hence, obtaining r_f^* is a discrete optimisation problem. This is, in general, an NP hard problem. Hence, we relax this constraint, and allow r_f to take any real value in $[\underline{\lambda}_f, \overline{\lambda}_f]$. The Lagrangian function for the rate optimisation problem is thus

$$\begin{aligned}
& L(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{u}, \boldsymbol{v}) \\
&= \sum_{f\in\mathcal{F}} \ln(1 - e_f) - \sum_{c\in\mathcal{C}} p_c\left(\sum_{f\in\mathcal{F}_c} \frac{k_f}{r_f w_{f,c}} - T_c\right) \\
&+ \sum_{f\in\mathcal{F}} u_f\left(r_f - \underline{\lambda}_f\right) - \sum_{f\in\mathcal{F}} v_f\left(r_f - \overline{\lambda}_f\right)
\end{aligned}$$

Applying KKT condition, $\frac{\partial L}{\partial r_f}|_{r_f^*} = 0$, we have

$$\begin{aligned}
\frac{-1}{1-e_f}\frac{\partial e_f}{\partial r_f}|_{r_f^*} &= \sum_{c\in\mathcal{C}_f} p_c \frac{-k_f}{r_f^{*2} w_{f,c}} + v_f - u_f \\
&= \frac{-k_f}{r_f^{*2}}\left(\sum_{c\in\mathcal{C}_f}\frac{p_c}{w_{f,c}}\right) + v_f - u_f \\
\frac{e_f}{1-e_f}\cdot\theta_f^* &= \frac{k_f}{r_f^{*2}}\left(\sum_{c\in\mathcal{C}_f}\frac{p_c}{w_{f,c}}\right) + v_f - u_f.
\end{aligned}$$

If the optimum r_f^* is either $\underline{\lambda}_f$ or $\overline{\lambda}_f$, then it is unique. If $r_f^* \in (\underline{\lambda}_f, \overline{\lambda}_f)$, then $u_f = v_f = 0$, which is the most interesting case, and we consider only this case for the rest of the paper.

Let $\lambda_f := \sum_{c \in \mathcal{C}_f} \frac{p_c}{w_{f,c}}$. The above equation becomes

$$\frac{e_f}{1-e_f} \cdot \theta_f^* = \frac{\lambda_f k_f}{r_f^{*2}} \quad (13)$$

$$e_f = \frac{\lambda_f k_f}{\lambda_f k_f + \theta_f^* r_f^{*2}} \quad (14)$$

$$\exp\left(-D_f D(\mathcal{B}(1-r_f^*) \| \mathcal{B}(\beta_f))\right) = \frac{\lambda_f k_f}{\lambda_f k_f + \theta_f^* r_f^{*2}}$$

$$D_f D(\mathcal{B}(1-r_f^*) \| \mathcal{B}(\beta_f)) = \ln\left(\frac{\lambda_f k_f + \theta_f^* r_f^{*2}}{\lambda_f k_f}\right) \quad (15)$$

In the above equation, the LHS is a strictly convex decreasing function of r_f^*. Since, the utility maximisation problem is a concave maximisation problem, the optimum rate $r_f^* \in (0, 1-\beta_f)$ exists and is unique.

E. Sub–gradient Approach to Compute optimum p_c^*

In this section, we discuss the procedure to obtain the Shadow costs or the Lagrange variables $\boldsymbol{p}^*$. The dual problem for the primal problem defined in Eqn. (11) is given by

$$\min_{\boldsymbol{p} \geq 0} \quad D(\boldsymbol{p}),$$

where the dual function $D(\boldsymbol{p})$ is given by

$$\begin{aligned} &D(\boldsymbol{p}) \\ &= \max_{\boldsymbol{r}} \sum_{f \in \mathcal{F}} \ln(1-e_f(r_f)) + \sum_{c \in \mathcal{C}} p_c \left(T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f w_{f,c}}\right) \quad (16) \\ &= \sum_{f \in \mathcal{F}} \ln(1-e_f(r_f^*(\boldsymbol{p}))) + \sum_{c \in \mathcal{C}} p_c \left(T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f^*(\boldsymbol{p}) w_{f,c}}\right). \quad (17) \end{aligned}$$

In the above equation, $e_f(r_f)$ denotes $e_f(\boldsymbol{\phi}_f^*, \theta_f^*(r_f), r_f)$. Since the dual function (of a primal problem) is convex, D is convex in $\boldsymbol{p}$. Hence, we use a sub–gradient method to obtain the optimum $\boldsymbol{p}^*$. From Eqn. (16), it is clear that for any $\boldsymbol{r}$,

$$D(\boldsymbol{p}) \geq \sum_{f \in \mathcal{F}} \ln(1-e_f(r_f)) + \sum_{c \in \mathcal{C}} p_c \left(T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f w_{f,c}}\right),$$

and in particular, $D(\boldsymbol{p})$ is greater than that for $r = r_f^*(\widetilde{\boldsymbol{p}})$, i.e.,

$$\begin{aligned} &D(\boldsymbol{p}) \\ &\geq \sum_{f \in \mathcal{F}} \ln(1-e_f(r_f^*(\widetilde{\boldsymbol{p}}))) + \sum_{c \in \mathcal{C}} p_c \left(T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f^*(\widetilde{\boldsymbol{p}}) w_{f,c}}\right) \\ &= D(\widetilde{\boldsymbol{p}}) + \sum_{c \in \mathcal{C}} (p_c - \widetilde{p}_c) \left(T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f^*(\widetilde{\boldsymbol{p}}) w_{f,c}}\right) \quad (18) \end{aligned}$$

Thus, a sub–gradient of $D(\cdot)$ at any $\widetilde{\boldsymbol{p}}$ is given by the vector

$$\left[T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f^*(\widetilde{\boldsymbol{p}}) w_{f,c}}\right]_{c \in \mathcal{C}}. \quad (19)$$

We obtain an iterative algorithm based on sub–gradient method that yields $\boldsymbol{p}^*$, with $\boldsymbol{p}(i)$ being the Lagrangians at the ith iteration.

$$p_c(i+1) = \left[p_c(i) - \gamma \cdot \left(T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f^*(\boldsymbol{p}(i)) w_{f,c}}\right)\right]^+.$$

where $\gamma > 0$ is a sufficiently small stepsize, and $[f(x)]^+ := \max\{f(x), 0\}$ ensures that the Lagrange multiplier never goes negative. Note that the Lagrangian updates can be locally done, as each cell c is required to know only the rates $r_f^*(\boldsymbol{p}(i))$ of flows $f \in \mathcal{F}_c$. Thus, at the beginning of each iteration i, the flows choose their coding rates to $r_f^*(\boldsymbol{p}(i))$, and each cell computes its cost based on the rates of flows through it. The updated costs along the route of each flow are then fed back to the source node to compute the rate for the next iteration.

The Lagrange multiplier p_c can be viewed as the cost of transmitting traffic through cell c. The amount of service time that is available is given by $\delta = T_c - \sum_{f \in \mathcal{F}_c} \frac{k_f}{r_f^*(\boldsymbol{p}(i)) w_{f,c}}$. When δ is positive and large, then the Lagrangian cost p_c decreases rapidly (because D is convex), and when δ is negative, then the Lagrangian cost p_c increases rapidly to make $\delta \geq 0$. We note that the increase or decrease of p_c between successive iterations is proportional to δ, the amount of service time available. Thus, the sub–gradient procedure provides a dynamic control scheme to balance the network loads.

We explore the properties of the optimum rate parameter r_f^* in Section IV-F. In Section V, we provide some examples that illustrate the optimum utility–fair resource allocation.

F. Properties of r_f^*

Lemma 3. *$r_f^*(D_f)$ is an increasing function of D_f.*

Lemma 3 is quite intuitive. For any given channel error β_f, as the deadline become less stringent, it is optimal to go for a high rate code. In other words, it is optimal for a flow to use as much scheduling time as possible (for a large D_f, and hence, use a high rate code); however, the resources are shared among multiple flows, and hence, we ask the following question: "*what is the optimal share of the scheduling time*" that each flow should have. Interestingly, in our problem formulation, the code rate r_f also solves this optimal scheduling times for each flows.

V. Examples

A. Example 1: Two cells with equal traffic load

We begin by considering the example shown in Figure 4 consisting of two cells a and b having three nodes 1, 2, and 3. Each cell has the same packet erasure probability β and the schedule length T. There are three flows f_1, f_2, and f_3, with two of the flows f_1 and f_3 having one–hop routes $\mathcal{C}_{f_1} = \{b\}$ and $\mathcal{C}_{f_3} = \{a\}$, and one flow f_2 having a two–hop route

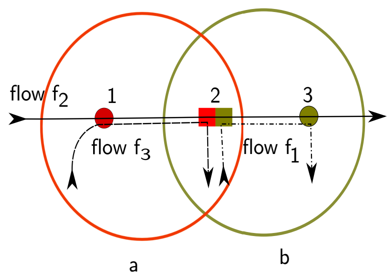

Fig. 4. Cells with equal traffic load

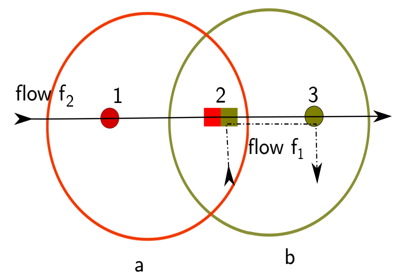

Fig. 5. Cells with unequal traffic load

$\mathcal{C}_{f_2} = \{a, b\}$. Each flow has the same information packet size k, decoding deadline D and PHY transmit rate, *i.e.* $w_{f,c} = w$. This is analogous to the so–called parking–lot topology often used to explore fairness issues.

The end–to–end erasure probability experienced by the two–hop flow f_2 is greater than that experienced by the one hop flows f_1 and f_3, since each hop has the same fixed erasure probability. Hence, we need to assign a lesser coding rate r_{f_2} to flow f_2 than to flows f_1 and f_3 in order to obtain the same error probability (after decoding) across flows. However, when operating at the boundary of the network capacity region (thereby maximising throughput), decreasing the coding rate r_{f_2} of the two–hop flow f_2 requires that the coding rate of *both* one–hop flows f_1 and f_3 be increased in order to remain within the available network capacity. In this sense, allocating coding rate to the two–hop flow f_2 imposes a greater marginal cost on the network (in terms of the sum–utility) than the one–hop flows, and we expect that a fair allocation will therefore assign higher coding rate to the two–hop flow f_2. The solution optimising this trade–off in a proportional fair manner can be understood using the analysis in the previous section.

In this example, both the cells are equally loaded and, by symmetry, the Lagrange multipliers $p_a = p_b$. Hence, $\lambda_{f_1} = \frac{\lambda_{f_2}}{2} = \lambda_{f_3}$. For the Chernoff–bound parameter $\boldsymbol{\theta} = [\theta, \theta]$, we find from Eqn. (13),

$$\begin{aligned} \frac{e_{f_2}}{1-e_{f_2}} \cdot \frac{1-e_{f_1}}{e_{f_1}} &= \frac{\lambda_{f_2}}{\lambda_{f_1}} \cdot \frac{r_{f_1}^{*2}}{r_{f_2}^{*2}} \\ &= 2 \cdot \frac{r_{f_1}^{*2}}{r_{f_2}^{*2}}. \end{aligned}$$

For sufficiently small erasure probabilities, we have

$$\begin{aligned} \frac{e_{f_2}}{e_{f_1}} &\approx 2 \cdot \frac{r_{f_1}^{*2}}{r_{f_2}^{*2}} \\ &\approx 2 \end{aligned}$$

Thus the proportional fair allocation is $e_{f_1} = e_{f_3} \approx 1/2 \cdot e_{f_2}$. That is, the coding rates are allocated such that the one–hop flows have approximately half the error probability of the two–hop flow.

B. Example 2: Two cells with unequal traffic load

We consider the same network as in the previous example, but now with only the flows f_1 and f_2 (i.e., the flow f_3 is not present) in the network. In this example, cell b carries two flows while cell a carries only one flow. The encoding rate constraints are given by

$$\begin{aligned} \frac{1}{r_{f_2}} &\leq \frac{wT}{k}, \text{ (from cell 1)}, \\ \frac{1}{r_{f_1}} + \frac{1}{r_{f_2}} &\leq \frac{wT}{k}, \text{ (from cell 2)}. \end{aligned}$$

Since, both r_{f_1} and r_{f_2} are at most 1, it is clear that at the optimal point, the rate constraint of cell a is not tight while the constraint of cell b is tight. Thus, the shadow prices (Lagrange multipliers) $p_a = 0$ and $p_b > 0$. That is, at the first hop the cell is not operating at capacity, and so the "price" for using this cell is zero. In this example, $\lambda_{f_1} = \lambda_{f_2}$, and hence, from Eqn. (13), we deduce that for sufficiently low cell erasure probability β, $e_{f_1} \approx e_{f_2}$. Alternatively, as the delay deadline $D \to \infty$, from Eqn. (13) we have $e_{f_1} = e_{f_2}$. These proportional fair allocations make sense intuitively since although flow f_2 crosses two hops, it is only constrained at the second hop and so it is natural to share the available capacity of this second hop approximately equally between the flows. When the erasure probability is sufficiently small, this yields approximately the same error probabilities for both flows. For larger erasure probabilities, it leads to the two–hop flow having higher error probability, in proportion to the per–hop erasure probability β.

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